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# SOLID CORES AND SOLID HULLS OF WEIGHTED BERGMAN SPACES.

JOSÉ BONET, WOLFGANG LUSKY, AND JARI TASKINEN

ABSTRACT. We determine the solid hull for  $2 < p < \infty$  and the solid core for  $1 < p < 2$  of weighted Bergman spaces  $A_\mu^p$ ,  $1 < p < \infty$ , of analytic functions on the disc and on the whole complex plane, for a very general class of non-atomic positive bounded Borel measures  $\mu$ . New examples are presented. Moreover we show that the space  $A_\mu^p$ ,  $1 < p < \infty$ , is solid if and only if the monomials are an unconditional basis of this space.

## 1. INTRODUCTION AND PRELIMINARIES.

Consider  $R = 1$  or  $R = \infty$  and  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ . We study holomorphic functions  $f : R \cdot \mathbb{D} \rightarrow \mathbb{C}$  where  $R \cdot \mathbb{D} = \mathbb{D}$  if  $R = 1$  and  $R \cdot \mathbb{D} = \mathbb{C}$  if  $R = \infty$ . Let  $\hat{f}(k)$  be the Taylor coefficients of  $f$ , i.e.  $f(z) = \sum_{k=0}^{\infty} \hat{f}(k)z^k$ . We take a non-atomic positive bounded Borel measure  $\mu$  on  $[0, R[$  such that  $\mu([r, R]) > 0$  for every  $r > 0$  and  $\int_0^R r^n d\mu(r) < \infty$  for all  $n > 0$ . Put, for  $1 \leq p < \infty$ ,

$$\|f\|_p = \left( \frac{1}{2\pi} \int_0^R \int_0^{2\pi} |f(re^{i\varphi})|^p d\varphi d\mu(r) \right)^{1/p}$$

and let

$$A_\mu^p = \{f : \mathbb{D} \rightarrow \mathbb{C} : f \text{ holomorphic with } \|f\|_p < \infty\}.$$

Let  $A$  be a vector space of holomorphic functions on  $R \cdot \mathbb{D}$  containing the polynomials. We want to study the *solid core*

$$s(A) = \{f \in A : g \in A \text{ for all holomorphic } g \text{ with } |\hat{g}(k)| \leq |\hat{f}(k)| \text{ for all } k\}$$

and the *solid hull*

$$S(A) = \{g : \mathbb{D} \rightarrow \mathbb{C} : g \text{ holomorphic, there is } f \in A \text{ with } |\hat{g}(k)| \leq |\hat{f}(k)| \text{ for all } k\}.$$

$A$  is called *solid* if  $A = S(A)$ .

In the first four sections we consider  $A = A_\mu^p$  while in section 5 we include the case where  $A$  consists of weighted sup-norm spaces of holomorphic functions.

The solid hull and core of spaces of analytic functions has been investigated by many authors. We refer the reader to the recent books [6] and [11] and the many references therein. For example in [6] the characterisation of the solid hulls and cores of  $A_\mu^p$  can be found where  $d\mu(r) = (1-r)^\alpha dr$  for some  $\alpha > 0$  and  $R = 1$ .

Originally, our main interest was to replace the “standard weights”  $(1-r)^\alpha$  by weights of the form  $v_{a,b}(r) = \exp(-a/(1-r)^b)$  for some  $a > 0$  and  $b > 0$ , which are of a completely different nature and require different methods, and hence to consider  $d\mu(r) = v_{a,b}(r)dr$ . We wanted to extend to weighted Bergman spaces the results of [3], a paper which was entirely devoted to this class of weights  $v_{a,b}$  in connection with weighted sup-norms. In the present article we give a characterization of solid hulls of  $A_\mu^p$  if  $2 < p < \infty$  and solid cores if  $1 < p < 2$  in Theorem 2.1 for much more

general  $\mu$  which, under some mild additional assumptions (Corollary 3.2), resulted in the explicit computation of many examples including  $v(r) = \exp(-a/(1-r)^b)$  for  $R = 1$  and  $v(r) = \exp(-r)$  for  $R = \infty$ ; see Corollaries 3.4 and 3.5.

The final sections 4 and 5 are dedicated to Bergman spaces and weighted sup-norm spaces which themselves are solid. We give examples for this situation in connection with holomorphic functions over the complex plane and show that this can never happen for holomorphic functions over the unit disc.

For a holomorphic  $g$  and  $0 < r$  we define

$$M_p(g, r) = \left( \frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\varphi})|^p d\varphi \right)^{1/p}$$

and  $P_n g(z) = \sum_{k=0}^n \hat{g}(k) z^k$ . It is well-known that, for  $1 < p < \infty$ , there are universal constants  $c_p > 0$  with  $M_p(P_n g, r) \leq c_p M_p(g, r)$  where  $c_p$  does not depend on  $g$ ,  $n$  or  $r$ . Moreover we have  $\lim_{n \rightarrow \infty} M_p(g - P_n g, r) = 0$ . Hence we obtain

$$\|P_n f\|_p \leq c_p \|f\|_p \text{ for all } f \in A_\mu^p \text{ and all } n \text{ and } \lim_{n \rightarrow \infty} \|f - P_n f\|_p = 0.$$

In particular we see that the monomials  $z \mapsto z^n$ ,  $n = 0, 1, 2, \dots$  form a Schauder basis of  $A_\mu^p$  if  $1 < p < \infty$ . Details can be seen in [4] and [12].

In the rest of the article  $[r]$  denotes the largest integer smaller or equal than  $r > 0$ .

## 2. MAIN GENERAL RESULT.

**Theorem 2.1.** *Assume that there are constants  $d_1, d_2 > 0$ , and  $\omega_n > 0, n = 1, 2, \dots$ , numbers  $0 \leq l_1 < l_2 < \dots$  and radii  $s_1 < s_2 < \dots$  such that, for every  $f \in A_\mu^p$ ,*

$$(2.1) \quad d_1 \|f\|_p \leq \left( \sum_{n=1}^{\infty} \omega_n^p M_p^p((P_{[l_{n+1}]} - P_{[l_n]})f, s_n) \right)^{1/p} \leq d_2 \|f\|_p.$$

(a) *If  $2 < p < \infty$ , then*

$$S(A_\mu^p) =$$

$$\left\{ g : R \cdot \mathbb{D} \rightarrow \mathbb{C} : g \text{ holomorphic with } \sum_{n=1}^{\infty} \omega_n^p \left( \sum_{k=[l_n]+1}^{[l_{n+1}]} |\hat{g}(k)|^2 s_n^{2k} \right)^{p/2} < \infty \right\}.$$

(b) *If  $1 < p < 2$ , then*

$$s(A_\mu^p) =$$

$$\left\{ g : R \cdot \mathbb{D} \rightarrow \mathbb{C} : g \text{ holomorphic with } \sum_{n=1}^{\infty} \omega_n^p \left( \sum_{k=[l_n]+1}^{[l_{n+1}]} |\hat{g}(k)|^2 s_n^{2k} \right)^{p/2} < \infty \right\}.$$

Theorem 2.1 is proved below. Before presenting the proof we point out that condition (2.1) can be realized for any given  $\mu$ . Indeed, fix  $\beta > 16 \cdot 3^{p-1} (1 + 2^p) c_p^p + 2$  and use induction to obtain  $0 = l_1 < l_2 < l_3 \dots$  and  $0 \leq s_1 < s_2 \dots < R$  with

$$(2.2) \quad \int_0^{s_n} r^{l_n p} d\mu = \beta \int_{s_n}^R r^{l_n p} d\mu \quad \text{and} \quad \int_0^{s_n} r^{l_{n+1} p} d\mu = \frac{1}{\beta} \int_{s_n}^R r^{l_{n+1} p} d\mu.$$

Instead of starting with  $n = 1$  we can as well start the induction e.g. with  $n = n_0$  for some  $n_0 \geq 0$  (with  $l_1 = 0$  and arbitrary  $s_1$ ) and restrict the preceding relations to all  $n \geq n_0$ . Moreover put

$$\omega_n = \left( \int_0^{s_n} \left( \frac{r}{s_n} \right)^{l_n p} d\mu + \int_{s_n}^R \left( \frac{r}{s_n} \right)^{l_{n+1} p} d\mu \right)^{1/p}.$$

Then there are constants  $d_1, d_2 > 0$  such that, for every  $f \in A_\mu^p$ ,

$$d_1 \|f\|_p \leq \left( \sum_{n=1}^{\infty} \omega_n^p M_p^p ((P_{[l_{n+1}]} - P_{[l_n]})f, s_n) \right)^{1/p} \leq d_2 \|f\|_p.$$

This was shown in [5] for  $p = 1$  and in [10] for  $1 < p < \infty$  and  $R = 1$ , but with some slight modifications the proofs carry over to the case  $R = \infty$ .

**Example 2.2.** (i) Let  $d\mu(r) = dr$  where  $R = 1$ . Then we obtain

$$l_n = \frac{1}{p}(a^{n-1} - 1) \text{ and } s_n = \left( \frac{\beta}{\beta + 1} \right)^{a^{1-n}} \text{ where } a = \frac{\log(\beta + 1)}{\log(1 + \beta) - \log(\beta)}.$$

This can be easily verified using the definition (starting with  $n = 0$ ) and induction.

(ii) Let  $d\mu(r) = r^\alpha dr$  for some  $\alpha > 0$  and  $R = 1$ . With example (i) and  $l_n p + \alpha = (a^{n-1} - 1)$ , where  $a$  is the number in (i), we obtain

$$l_n = \frac{1}{p}(a^{n-1} - 1) - \frac{\alpha}{p} \quad \text{and} \quad s_n = \left( \frac{\beta}{\beta + 1} \right)^{a^{1-n}}$$

for  $n \geq 2$  with  $l_1 = 0$  and  $s_1 = \beta/(\beta + 1)$ .

Now we turn to the proof of Theorem 2.1. Let  $f : R \cdot \mathbb{D} \rightarrow \mathbb{C}$  be holomorphic. Recall that  $\hat{f}(n)r^n = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\varphi})e^{-in\varphi} d\varphi$  for each  $0 < r < R$  and each  $n = 0, 1, 2, \dots$ . For  $g(re^{i\varphi}) = r^{n(p-1)}e^{-in\varphi}/(\int_0^R r^{np} d\mu)^{1-1/p}$  we have

$$|\hat{f}(n)| \left( \int_0^R r^{np} d\mu \right)^{1/p} = \frac{1}{2\pi} \left| \int_0^R \int_0^{2\pi} f(re^{i\varphi})g(re^{i\varphi})d\varphi d\mu \right| \leq \|f\|_p.$$

In the following we make use of the Khintchine inequality ([7], 2.b.3.), i.e. for arbitrary  $b_k$  and  $n$  we have

$$A_p \left( \sum_{k=1}^n |b_k|^2 \right)^{1/2} \leq \left( \frac{1}{2^n} \sum_{\theta_k = \pm 1} \left| \sum_{k=1}^n b_k \theta_k \right|^p \right)^{1/p} \leq B_p \left( \sum_{k=1}^n |b_k|^2 \right)^{1/2}$$

where  $A_p, B_p$  are universal constants not depending on  $n$ . (The summation in the central expression runs over the  $2^n$  different possibilities of the change of signs.)

**Conclusion of the proof of Theorem 2.1.** For a holomorphic function  $g$  put

$$\alpha(g) = \left( \sum_{n=1}^{\infty} \omega_n^p M_p^p ((P_{[l_{n+1}]} - P_{[l_n]})f, s_n) \right)^{1/p}.$$

As assumed,  $\alpha(\cdot)$  is equivalent to  $\|\cdot\|_p$ . Moreover let

$$\gamma(g) = \left( \sum_{n=1}^{\infty} \omega_n^p \left( \sum_{k=[l_n]+1}^{[l_{n+1}]} |\hat{g}(k)|^2 s_n^{2k} \right)^{p/2} \right)^{1/p}$$

and  $V = \{g : R \cdot \mathbb{D} \rightarrow \mathbb{C} : g \text{ holomorphic with } \gamma(g) < \infty\}$ . Recall that Parseval's identity implies

$$M_2^2((P_{[l_{n+1}]} - P_{[l_n]})f, s_n) = \sum_{k=[l_n]+1}^{[l_{n+1}]} |\hat{g}(k)|^2 s_n^{2k}.$$

**Proof of (a).** Let  $g \in S(A_\mu^p)$ . Then there is  $f \in A_\mu^p$  with  $|\hat{g}(k)| \leq |\hat{f}(k)|$  for all  $k$ . If  $2 < p < \infty$  then

$$\gamma(g) \leq \gamma(f) \leq \alpha(f) \leq d_2 \|f\|_p < \infty.$$

Hence  $g \in V$ .

Now let  $g \in V$ . Put  $\Delta_n = \{+1, -1\}^{[l_{n+1}]-[l_n]}$ . For  $\Theta_n = (\theta_{[l_n]+1}, \dots, \theta_{[l_{n+1}]}) \in \Delta_n$  put

$$g_{\Theta_n}(\varphi) = \sum_{k=[l_n]+1}^{[l_{n+1}]} \theta_k \hat{g}(k) s_n^k e^{ik\varphi} \text{ and } g_n(\varphi) = \sum_{k=[l_n]+1}^{[l_{n+1}]} \hat{g}(k) s_n^k e^{ik\varphi}.$$

Let  $\tilde{\Theta}_n$  be such that

$$M_p(g_{\tilde{\Theta}_n}, s_n) \leq \left( \frac{1}{2^{[l_{n+1}]-[l_n]}} \sum_{\Theta_n \in \Delta_n} M_p^p(g_{\Theta_n}, s_n) \right)^{1/p}.$$

The Khintchine inequality yields

$$M_p(g_{\tilde{\Theta}_n}, s_n) \leq B_p M_2(g_n, s_n).$$

Put  $h = \sum_n g_{\tilde{\Theta}_n}$ . Then, by the preceding estimates,

$$d_1 \|h\|_p \leq \alpha(h) \leq B_p \gamma(g) < \infty.$$

Hence  $h \in A_\mu^p$ . Since by definition  $|\hat{h}(k)| = |\hat{g}(k)|$  for all  $k$  we obtain  $g \in S(A_\mu^p)$ .

**Proof of (b).** We retain the preceding notation. Let  $g \in V$  and let  $f : R \cdot \mathbb{D} \rightarrow \mathbb{C}$  be holomorphic with  $|\hat{f}(k)| \leq |\hat{g}(k)|$  for all  $k$ . Then

$$d_1 \|f\|_p \leq \alpha(f) \leq \gamma(f) \leq \gamma(g) < \infty.$$

This implies  $f \in A_\mu^p$  and hence  $g \in s(A_\mu^p)$ .

Now let  $g \in s(A_\mu^p)$ . Let  $\tilde{\tilde{\Theta}}_n \in \Delta_n$  be such that

$$\left( \frac{1}{2^{[l_{n+1}]-[l_n]}} \sum_{\Theta_n \in \Delta_n} M_p^p(g_{\Theta_n}, s_n) \right)^{1/p} \leq M_p(g_{\tilde{\tilde{\Theta}}_n}, s_n).$$

Put  $h = \sum_n g_{\tilde{\tilde{\Theta}}_n}$ . Then we obtain  $|\hat{h}(k)| = |\hat{g}(k)|$  for all  $k$ . Hence  $h \in A_\mu^p$ . The

Khintchine inequality together with the choice of  $\tilde{\tilde{\Theta}}_n$  yields

$$\gamma(g) = \gamma(h) \leq A_p^{-1} \alpha(h) \leq d_2 A_p^{-1} \|h\|_p < \infty.$$

We conclude  $g \in V$ .  $\square$

## 3. MAIN EXAMPLES.

Quite often it is very difficult to compute the parameters  $l_n$  and  $s_n$  in (2.2). Therefore it is worthwhile to consider special cases which yield an equivalent representation of the norm  $\|\cdot\|_p$  satisfying (2.1) and which are easier to compute and cover many examples. To this end let  $v : [0, R[ \rightarrow ]0, \infty[$  be a weight function, i.e. let  $v$  be continuous, decreasing and satisfy

$$\lim_{r \rightarrow R} v(r) = 0 \quad \text{and} \quad \sup_r r^n v(r) < \infty \quad \text{for all } n > 0.$$

Moreover, let  $\nu$  be a non-atomic positive Borel measure on  $[0, R[$  such that  $\nu([r, R]) > 0$  for every  $r > 0$ , and  $\int_0^R r^n v(r) d\nu(r) < \infty$  for every  $n \geq 0$ . Put, for  $1 \leq p < \infty$ ,

$$\|f\|_p = \left( \int_0^R M_p^p(f, r) v(r) d\nu(r) \right)^{1/p}$$

Here we consider  $A_\mu^p$  with  $d\mu(r) = v(r) d\nu(r)$ . Actually one can relax a bit the conditions on  $v$ . It suffices to require that  $v$  be decreasing on  $[r_0, R[$  for some  $r_0 \in ]0, R[$ . This follows from the fact that, for  $d\tilde{\mu} = 1_{[r_0, R[} d\mu$ , the  $L_p$ -norms with respect to  $\mu$  and  $\tilde{\mu}$  are equivalent. Actually, using the fact that  $M_p(f, r)$  is increasing with respect to  $r$  for holomorphic functions  $f$  we see that

$$\int_{r_0}^R M_p^p(f, r) d\mu(r) \leq \int_0^R M_p^p(f, r) d\mu(r) \leq \left( 1 + \frac{\mu([r_0, R])}{\mu([0, R])} \right) \int_{r_0}^R M_p^p(f, r) d\mu(r).$$

For any  $n > 0$  let  $r_n \in [0, R[$  be a point where the function  $r \mapsto r^n v(r)$  attains its global maximum. It is easily seen that  $r_m < r_n$  if  $m < n$ . In the following we want to assume that

$$(3.1) \quad \begin{aligned} & r_n \text{ is the unique global maximum of } r^n v(r) \text{ for all } n \\ & \text{and there are no further local maxima.} \end{aligned}$$

For example this is the case if  $v$  is differentiable and  $v'/v$  is injective. The assumption (3.1) implies that  $r^n v(r)$  is decreasing for  $r \geq r_n$ . Moreover we assume that  $v$  satisfies

**Condition  $(b_0)$ :** There are numbers  $1 < b < K$  and  $m_1 < m_2 < \dots$  with  $\lim_{n \rightarrow \infty} m_n = \infty$  such that

$$b \leq \left( \frac{r_{m_n}}{r_{m_{n+1}}} \right)^{m_n} \frac{v(r_{m_n})}{v(r_{m_{n+1}})}, \left( \frac{r_{m_{n+1}}}{r_{m_n}} \right)^{m_{n+1}} \frac{v(r_{m_{n+1}})}{v(r_{m_n})} \leq K.$$

Condition  $(b_0)$  is exactly the same as condition (b) in [3], except that the treatment of weighted Banach spaces of analytic functions with sup-norms requires  $2 < b < K$ . We refer the reader to [3] and [9] for more information and examples related to these conditions.

We take the parameters of condition  $(b_0)$  and we put

$$I_n = \nu([r_{m_n}, r_{m_{n+1}}])$$

and assume

$$(3.2) \quad I_n < \infty \quad \text{for all } n \quad \text{and} \quad \limsup_{n \rightarrow \infty} \frac{I_n}{\min(I_{n-1}, I_{n+1})} < b.$$

**Theorem 3.1.** *Let  $1 < p < \infty$ . Assume that  $v$  satisfies  $(b_0)$  with (3.1), (3.2). Then there are constants  $d_1, d_2 > 0$  with*

$$(3.3) \quad d_1 \|f\|_p \leq \left( \sum_{n=1}^{\infty} M_p^p((P_{[m_{n+1}/p]} - P_{[m_n/p]})f, r_{m_n})v(r_{m_n})I_n \right)^{1/p} \leq d_2 \|f\|_p.$$

for all  $f \in A_\mu^p$ .

In view of (2.1) we can apply Theorem 2.1 with the preceding  $l_n = m_n/p$ ,  $\omega_n^p = v(r_{m_n})I_n$  and  $s_n = r_{m_n}$ .

**Corollary 3.2.** *Let  $d\mu = v dv$ .*

(a) *If  $2 < p < \infty$ , then*

$$S(A_\mu^p) = \{g : R \cdot \mathbb{D} \rightarrow \mathbb{C} :$$

$$g \text{ holomorphic with } \sum_{n=1}^{\infty} v(r_{m_n})I_n \left( \sum_{k=[m_n/p]+1}^{[m_{n+1}/p]} |\hat{g}(k)|^2 r_{m_n}^{2k} \right)^{p/2} < \infty\}.$$

(b) *If  $1 < p < 2$ , then*

$$s(A_\mu^p) = \{g : R \cdot \mathbb{D} \rightarrow \mathbb{C} :$$

$$g \text{ holomorphic with } \sum_{n=1}^{\infty} v(r_{m_n})I_n \left( \sum_{k=[m_n/p]+1}^{[m_{n+1}/p]} |\hat{g}(k)|^2 r_{m_n}^{2k} \right)^{p/2} < \infty\}.$$

Before we prove Theorem 3.1 we present the following examples. They are concrete cases to which Corollary 3.2 applies, thus permitting us to calculate explicitly all the parameters which appear in the solid hull and solid core.

**Example 3.3.** (i)  $R = 1$  and  $d\mu(r) = \exp(-\alpha/(1-r)^\beta)dr$  for some  $\alpha, \beta > 0$ . We take  $v(r) = \exp(-\alpha/(1-r)^\beta)$  and  $d\nu(r) = dr$ .  $v$  satisfies condition  $(b_0)$  with

$$m_n = \beta \left( \frac{\beta}{\alpha} \right)^{1/\beta} n^{2+2/\beta} - \beta n^2 \quad \text{and} \quad r_{m_n} = 1 - \left( \frac{\alpha}{\beta} \right)^{1/\beta} \frac{1}{n^{2/\beta}}$$

and  $b = e^1$  (see [3], Theorem 3.1.) Here  $I_n = (\alpha/\beta)^{1/\beta} (n^{-2/\beta} - (n+1)^{-2/\beta})$ . Hence

$$\lim_{n \rightarrow \infty} \frac{I_n}{\min(I_{n-1}, I_{n+1})} = 1.$$

This shows that (3.2) is satisfied. (3.1) holds, too, according to [3]. So we can apply Corollary 3.2.

(ii)  $R = 1$  and  $d\mu(r) = (1 - \log(1-r))^{-1}dr$ . Here we take

$$v(r) = 1 - r \quad \text{and} \quad d\nu(r) = \frac{dr}{(1-r)(1 - \log(1-r))}.$$

$r_m = 1 - 1/(m+1)$  is the only zero of the derivative of  $r^m v(r)$ . Hence (3.1) is satisfied. If we take  $m_n = 9^n$  and hence  $r_{m_n} = 1 - 1/(9^n + 1)$  then a simple calculation reveals that  $v$  satisfies  $(b_0)$  with  $b = 3$ . We obtain

$$I_n = \int_{r_{m_n}}^{r_{m_{n+1}}} d\nu = \log \left( \frac{1 + \log(9^{n+1} + 1)}{1 + \log(9^n + 1)} \right)$$

from which we infer  $\lim_{n \rightarrow \infty} I_n / \min(I_{n-1}, I_{n+1}) = 1$ . This implies (3.2).

(iii)  $R = \infty$  and  $d\mu(r) = e^{-r}dr$ . Here we take  $v(r) = e^{-r}$ ,  $d\nu(r) = dr$ .  $r_m = m$  is the unique zero of the derivative of  $r^m v(r)$ . Hence (3.1) is satisfied. Put

$$m_1 = 1 \quad \text{and} \quad m_{n+1} = m_n + 2\sqrt{m_n}, \quad n = 1, 2, \dots, \quad \text{and} \quad r_{m_n} = m_n.$$

A simple calculation yields, with

$$-x - \frac{1}{2} \left( \frac{x}{1-x} \right)^2 \leq \log(1-x) \leq -x \quad \text{if } 0 < x < 1,$$

$$\begin{aligned} \exp \left( \frac{4\sqrt{m}}{\sqrt{m}+2} - 2 \right) &\leq \left( \frac{r_{m_n}}{r_{m_{n+1}}} \right)^{m_n} \frac{v(r_{m_n})}{v(r_{m_{n+1}})} = \\ &\exp \left( m \log \left( 1 - \frac{2}{\sqrt{m}+2} \right) + 2\sqrt{m} \right) \leq \exp \left( \frac{4\sqrt{m}}{\sqrt{m}+2} \right). \end{aligned}$$

Similarly, with

$$x - \frac{x^2}{2} \leq \log(1+x) \leq x \quad \text{for } 0 < x < 1,$$

$$\begin{aligned} \exp \left( 4 - 2 \left( 1 + \frac{2}{\sqrt{m}} \right) \right) &\leq \exp \left( (m + 2\sqrt{m}) \log \left( 1 + \frac{2}{\sqrt{m}} \right) - 2\sqrt{m} \right) \\ &= \left( \frac{r_{m_{n+1}}}{r_{m_n}} \right)^{m_{n+1}} \frac{v(r_{m_{n+1}})}{v(r_{m_n})} \leq e^4. \end{aligned}$$

This shows that condition  $(b_0)$  holds. Moreover we easily obtain

$$I_n = 2\sqrt{m_n} \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{I_n}{\min(I_{n-1}, I_{n+1})} = 1$$

which yields (3.2). Observe that in this case we can take  $m_n = n^2$ ; see Theorem 3.1 in [1]. This fact is not surprising, since one can easily prove by induction that our selection of  $m_n$  above satisfies  $(n-1)^2 \leq m_n \leq n^2$  for each  $n$ .

**Corollary 3.4.** *Let  $R = 1$  and  $d\mu(r) = \exp(-1/(1-r))dr$ .*

(a) *If  $2 < p < \infty$ , then*

$$S(A_\mu^p) = \{g \in H(\mathbb{D}) :$$

$$\sum_{n=1}^{\infty} e^{-n^2} \left( \frac{1}{n^2} - \frac{1}{(n+1)^2} \right) \left( \sum_{k=[n^4/p]+1}^{[(n+1)^4/p]} |\hat{g}(k)|^2 \left( 1 - \frac{1}{n^2} \right)^{2k} \right)^{p/2} < \infty \}.$$

(b) *If  $1 < p < 2$ , then*

$$s(A_\mu^p) = \{g \in H(\mathbb{D}) :$$

$$\sum_{n=1}^{\infty} e^{-n^2} \left( \frac{1}{n^2} - \frac{1}{(n+1)^2} \right) \left( \sum_{k=[n^4/p]+1}^{[(n+1)^4/p]} |\hat{g}(k)|^2 \left( 1 - \frac{1}{n^2} \right)^{2k} \right)^{p/2} < \infty \}.$$

*Proof.* Example 3.3 (i) in [3] shows that we can take, for  $v(r) = \exp(-1/(1-r))$ ,  $m_n = n^4$  for each  $n$ . The result follows from Example 3.3 (i) and Corollary 3.2.  $\square$



**Corollary 3.5.** *Let  $R = \infty$  and  $d\mu(r) = e^{-r}dr$ .*

(a) *If  $2 < p < \infty$ , then*

$$S(A_\mu^p) = \{g \in H(\mathbb{C}) : \sum_{n=1}^{\infty} e^{-n^2} 2n \left( \sum_{k=[n^2/p]+1}^{[(n+1)^2/p]} |\hat{g}(k)|^2 n^{2k} \right)^{p/2} < \infty\}.$$

(b) *If  $1 < p < 2$ , then*

$$s(A_\mu^p) = \{g \in H(\mathbb{C}) : \sum_{n=1}^{\infty} e^{-n^2} 2n \left( \sum_{k=[n^2/p]+1}^{[(n+1)^2/p]} |\hat{g}(k)|^2 n^{2k} \right)^{p/2} < \infty\}.$$

*Proof.* It is a consequence of Example 3.3 (iii) and Corollary 3.2.  $\square$

**Lemma 3.6.** *Let  $1 \leq p < \infty$ ,  $0 < r < s$  and  $f(z) = \sum_{m \leq j \leq n} \alpha_j z^j$  for some  $\alpha_j$  and  $0 \leq m < n$ . Then we have*

$$(i) \quad M_p(f, r) \leq \left(\frac{r}{s}\right)^m M_p(f, s)$$

and

$$(ii) \quad M_p(f, s) \leq \left(\frac{s}{r}\right)^n M_p(f, r).$$

*Proof.* Part (i) follows from the fact that, for holomorphic  $f$ , the function  $M_p(f, \cdot)$  is increasing in  $r$  while (ii) is Lemma 3.1. (i) of [8].  $\square$

Now consider  $1 < p < \infty$  and let  $m_n, I_n$  satisfy (b<sub>0</sub>) and (3.1), (3.2).

**Lemma 3.7.** *Fix  $k, n$  and  $r_{m_k} \leq r \leq r_{m_{k+1}}$ . Then we have*

$$(i) \quad \left(\frac{r}{r_{m_n}}\right)^{m_n} \frac{v(r)}{v(r_{m_n})} \leq \left(\frac{1}{b}\right)^{n-k-1} \quad \text{if } k < n$$

and

$$(ii) \quad \left(\frac{r}{r_{m_n}}\right)^{m_{n+1}} \frac{v(r)}{v(r_{m_n})} \leq K \left(\frac{1}{b}\right)^{k-n-1} \quad \text{if } k \geq n.$$

*Proof.* If  $k < n$  we have

$$\begin{aligned} \left(\frac{r}{r_{m_n}}\right)^{m_n} \frac{v(r)}{v(r_{m_n})} &= \\ \left(\frac{r}{r_{m_{k+1}}}\right)^{m_n} \frac{v(r)}{v(r_{m_{k+1}})} \left(\frac{r_{m_{k+1}}}{r_{m_{k+2}}}\right)^{m_n} \frac{v(r_{m_{k+1}})}{v(r_{m_{k+2}})} \cdots \left(\frac{r_{m_{n-1}}}{r_{m_n}}\right)^{m_n} \frac{v(r_{m_{n-1}})}{v(r_{m_n})} &\leq \\ \left(\frac{r}{r_{m_{k+1}}}\right)^{m_{k+1}} \frac{v(r)}{v(r_{m_{k+1}})} \left(\frac{r_{m_{k+1}}}{r_{m_{k+2}}}\right)^{m_{k+2}} \frac{v(r_{m_{k+1}})}{v(r_{m_{k+2}})} \cdots \left(\frac{r_{m_{n-1}}}{r_{m_n}}\right)^{m_n} \frac{v(r_{m_{n-1}})}{v(r_{m_n})} & \\ \leq \left(\frac{1}{b}\right)^{n-k-1} & \end{aligned}$$

If  $k \geq n+1$  we have

$$\begin{aligned} \left(\frac{r}{r_{m_n}}\right)^{m_{n+1}} \frac{v(r)}{v(r_{m_n})} &= \\ \left(\frac{r}{r_{m_k}}\right)^{m_{n+1}} \frac{v(r)}{v(r_{m_k})} \left(\frac{r_{m_k}}{r_{m_{k-1}}}\right)^{m_{n+1}} \frac{v(r_{m_k})}{v(r_{m_{k-1}})} \cdots \left(\frac{r_{m_{n+1}}}{r_{m_n}}\right)^{m_{n+1}} \frac{v(r_{m_{n+1}})}{v(r_{m_n})} &\leq \end{aligned}$$

$$\begin{aligned} & \left( \frac{r}{r_{m_k}} \right)^{m_k} \frac{v(r)}{v(r_{m_k})} \left( \frac{r_{m_k}}{r_{m_{k-1}}} \right)^{m_{k-1}} \frac{v(r_{m_k})}{v(r_{m_{k-1}})} \cdots \left( \frac{r_{m_{n+2}}}{r_{m_{n+1}}} \right)^{m_{n+1}} \frac{v(r_{m_{n+1}})}{v(r_{m_n})} K \\ & \leq K \left( \frac{1}{b} \right)^{k-n-1} \end{aligned}$$

Similarly, for  $k = n$ ,

$$\left( \frac{r}{r_{m_n}} \right)^{m_{n+1}} \frac{v(r)}{v(r_{m_n})} \leq \left( \frac{r_{m_{n+1}}}{r_{m_n}} \right)^{m_{n+1}} \frac{v(r_{m_{n+1}})}{v(r_{m_n})} \leq K.$$

□

Now fix  $k_0 > 0$  and  $0 < \rho < b$  such that

$$(3.4) \quad \frac{I_n}{\min(I_{n-1}, I_{n+1})} \leq \rho \quad \text{if } k \geq k_0.$$

**Corollary 3.8.** *Let  $f_n(z) = \sum_{m_n/p \leq j < m_{n+1}/p} \alpha_j z^j$  where  $n \geq k_0$ . Then, for any  $k \geq k_0$  we have*

$$(3.5) \quad \int_{r_{m_k}}^{r_{m_{k+1}}} M_p^p(f_n, r) v(r) d\nu(r) \leq c \left( \frac{\rho}{b} \right)^{|n-k|} M_p^p(f_n, r_{m_n}) v(r_{m_n}) I_n.$$

Here  $c > 0$  is a universal constant independent of  $k, n, f_n$ .

*Proof.* First let  $k < n$ . Then Lemma 3.6 (i) and Lemma 3.7 (i) imply

$$\begin{aligned} & \int_{r_{m_k}}^{r_{m_{k+1}}} M_p^p(f_n, r) v(r) d\nu(r) \\ & \leq M_p^p(f_n, r_{m_n}) v(r_{m_n}) \int_{r_{m_k}}^{r_{m_{k+1}}} \left( \frac{r}{r_{m_n}} \right)^{m_n} \frac{v(r)}{v(r_{m_n})} d\nu(r) \\ & \leq c_0 M_p^p(f_n, r_{m_n}) v(r_{m_n}) I_n \left( \prod_{j=k}^{n-1} \frac{I_j}{I_{j+1}} \right) \left( \frac{1}{b} \right)^{|n-k|} \\ & \leq c_1 \left( \frac{\rho}{b} \right)^{|n-k|} M_p^p(f_n, r_{m_n}) v(r_{m_n}) I_n, \end{aligned}$$

where  $c_0, c_1$  are universal constants. If  $k \geq n$  then we use Lemma 3.6 (ii) and Lemma 3.7 (ii) to get

$$\begin{aligned} & \int_{r_{m_k}}^{r_{m_{k+1}}} M_p^p(f_n, r) v(r) d\nu(r) \\ & \leq M_p^p(f_n, r_{m_n}) v(r_{m_n}) \int_{r_{m_k}}^{r_{m_{k+1}}} \left( \frac{r}{r_{m_n}} \right)^{m_{n+1}} \frac{v(r)}{v(r_{m_n})} d\nu(r) \\ & \leq K b M_p^p(f_n, r_{m_n}) v(r_{m_n}) I_n \left( \prod_{j=n}^{k-1} \frac{I_{j+1}}{I_j} \right) \left( \frac{1}{b} \right)^{|n-k|} \\ & \leq c_2 \left( \frac{\rho}{b} \right)^{|n-k|} M_p^p(f_n, r_{m_n}) v(r_{m_n}) I_n, \end{aligned}$$

where  $c_2$  is a universal constant.

□

**Conclusion of the proof of Theorem 3.1** Let  $f \in A_\mu^p$ , say  $f = \sum_n f_n$  where  $f_n$  is as in Corollary 3.8. We can assume that  $f_n = 0$  for  $n \leq k_0$  with  $k_0$  as in (3.4).

To prove the right-hand inequality in Theorem 3.1 we use that  $M_p(f_n, r) \leq cM_p(f, r)$  for a universal constant independent of  $r$ , as well as that, in view of (3.1),  $r^{m_n}v(r)$  is decreasing for  $r \geq r_{m_n}$ . We have

$$\begin{aligned}
& \sum_n M_p^p(f_n, r_{m_n})v(r_{m_n})I_n \\
& \leq \sum_n \int_{r_{m_n}}^{r_{m_n+1}} \left(\frac{r_{m_n}}{r}\right)^{m_n} \frac{v(r_{m_n})}{v(r)} M_p^p(f_n, r)v(r) d\nu(r) \\
& \leq \sum_n \int_{r_{m_n}}^{r_{m_n+1}} \left(\frac{r_{m_n}}{r_{m_n+1}}\right)^{m_n} \frac{v(r_{m_n})}{v(r_{m_n+1})} M_p^p(f_n, r)v(r) d\nu(r) \\
& \leq K \sum_n \int_{r_{m_n}}^{r_{m_n+1}} M_p^p(f_n, r)v(r) d\nu(r) \\
& \leq c^p K \sum_n \int_{r_{m_n}}^{r_{m_n+1}} M_p^p(f, r)v(r) d\nu(r) \\
& \leq c^p K \|f\|_p^p.
\end{aligned}$$

This in particular implies that  $\sum_n M_p^p(f_n, r_{m_n})v(r_{m_n})I_n < \infty$ .

Now we show the left-hand inequality of Theorem 3.1. Using the Minkowski inequality in the first estimate and Corollary 3.8 in the second one, we obtain

$$\begin{aligned}
\|f\|_p^p &= \sum_k \int_{r_{m_k}}^{r_{m_{k+1}}} M_p^p(f, r)v(r) d\nu(r) \\
&\leq \sum_k \left( \sum_n \left( \int_{r_{m_k}}^{r_{m_{k+1}}} M_p^p(f_n, r)v(r) d\nu(r) \right)^{1/p} \right)^p \\
&\leq c_1 \sum_k \left( \sum_n \left( \frac{\rho}{b} \right)^{|n-k|/p} (M_p^p(f_n, r_{m_n})v(r_{m_n})I_n)^{1/p} \right)^p \\
&\leq c_2 \sum_k \sum_n \left( \frac{\rho}{b} \right)^{|n-k|/p} M_p^p(f_n, r_{m_n})v(r_{m_n})I_n \\
&\leq c_3 \sum_n M_p^p(f_n, r_{m_n})v(r_{m_n})I_n.
\end{aligned}$$

Here  $c_1, c_2, c_3$  are universal constants. In the second last inequality we used the Hölder inequality in the following way: Put  $a_n = (M_p^p(f_n, r_{m_n})v(r_{m_n})I_n)^{1/p}$ . Then

$$\sum_n \left( \frac{\rho}{b} \right)^{|n-k|/p} a_n \leq \left( \sum_n \left( \frac{\rho}{b} \right)^{|n-k|/p} a_n^p \right)^{1/p} \cdot \left( \sum_n \left( \frac{\rho}{b} \right)^{|n-k|/p} \right)^{1/q},$$

with  $1/p + 1/q = 1$ . In the last inequality we interchanged the summation over  $k$  and  $n$  and utilized  $\sup_k \sum_n (\rho/b)^{|n-k|/p} = \sup_n \sum_k (\rho/b)^{|n-k|/p} < \infty$ .  $\square$

#### 4. SOLID BERGMAN SPACES.

Recall, a Bergman space  $A_\mu^p$  is solid if  $S(A_\mu^p) = A_\mu^p$ .

**Theorem 4.1.** *Let  $1 < p < \infty, p \neq 2$ . Then the following are equivalent*

- (i)  $A_\mu^p$  is solid
- (ii)  $s(A_\mu^p) = A_\mu^p$
- (iii) The monomials  $(z^n)_{n=0}^\infty$  are an unconditional basis of  $A_\mu^p$
- (iv) The normalized monomials  $(z^n/\|z^n\|_p)_{n=0}^\infty$  are equivalent to the unit vector basis of  $l^p$
- (v)  $\sup_n(l_{n+1} - l_n) < \infty$  for the numbers  $l_n$  in (2.1)

**Remark 4.2.** If  $p = 2$  then the normalized monomials are an orthonormal basis for  $A_\mu^2$  and all conditions (i)-(iv) are satisfied.

The following example is relevant in connection with Theorem 4.1.

**Example.** Consider  $R = \infty$  and  $v(r) = \exp(-\log^2(r))$ ,  $d\nu(r) = dr$ . (This is included in Example 2.2 of [9].)  $v$  is decreasing on  $[1, \infty[$  which suffices in view of the remarks in the beginning of section 3. We easily see that  $r_m = \exp(m/2)$  is the only zero of the derivative of  $r^m v(r)$ . Hence (3.1) is satisfied. We get for any  $n > 0$  and  $m > 0$

$$\left(\frac{r_m}{r_n}\right)^m \frac{v(r_m)}{v(r_n)} = \left(\frac{r_n}{r_m}\right)^n \frac{v(r_n)}{v(r_m)} = \exp\left(\frac{(n-m)^2}{4}\right).$$

So, if we take  $m_n = 4n$  then condition  $(b_0)$  is satisfied with  $b = e^4$ . Moreover we have  $I_n = \exp(2n+2) - \exp(2n)$ . An easy calculation shows that (3.2) holds. Hence we can consider (2.1) with  $l_n = m_n/p$ . Therefore  $\sup_n(l_{n+1} - l_n) = 4/p < \infty$ . This means, for  $d\mu(r) = v(r)dr$ , the Bergman space  $A_\mu^p$  is solid.

For the preceding example it is essential that  $R = \infty$ . Indeed, we have

**Corollary 4.3.** *Let  $1 < p < \infty, p \neq 2$ , and  $R = 1$ . Then no Bergman space  $A_\mu^p$  is solid.*

We prove Corollary 4.3 at the end of this section. For the proof of Theorem 4.1 we need the following

**Lemma 4.4.** *Let  $(e_n)$  be a Schauder basis of a Banach space  $X$  with basis projections  $P_n$ . For  $M \subset \mathbb{N}$ , let  $T_M$  be the linear (not necessarily continuous) operator defined in the linear span of  $(e_n)$  by  $T_M e_k = e_k$  if  $k \in M$  and  $T_M e_k = 0$  otherwise.*

*If the basis  $(e_n)$  is not unconditional, then there is  $N \subset \mathbb{N}$  such that, for any  $n$ , there exists  $m_n$  and  $0 \neq y \in P_{m_n} X$  with  $\|y\| \leq n \|T_N y\|$ .*

*Proof.* If  $(e_n)$  is a conditional basis then there exists an operator of the form  $T_N$  which is unbounded on  $X$ . Hence there is a sequence  $x_n \in X$  with  $\|x_k\| = 1$  and  $\lim_{k \rightarrow \infty} \|T_N x_k\| = \infty$ . For suitable  $m_n$  we find  $k_n$  such that  $0 < \|P_{m_n} x_{k_n}\| \leq \|T_N P_{m_n} x_{k_n}\|$ . Here we use  $P_{m_n} T_N = T_N P_{m_n}$ .  $\square$

In the following we retain the definition of  $T_N$  with respect to the monomials  $(z^n)$ .

**Lemma 4.5.** *Let  $1 < p < \infty, p \neq 2$  and assume that there are constants  $c_n > 0$ ,  $d_n > 0$  with  $\sup_n d_n/c_n < \infty$ , integers  $0 < a_n < b_n < a_{n+1}$  and radii  $s_n$  such that, for any  $f_n \in A_\mu^p$  with  $f_n(z) = \sum_{a_n \leq j \leq b_n} \alpha_j z^j$  we have*

$$c_n M_p(f_n, s_n) \leq \|f_n\|_p \leq d_n M_p(f_n, s_n).$$

*If  $\sup_n(b_n - a_n) = \infty$  then the monomials are not unconditional in  $A_\mu^p$ .*

*Proof.* It is well known that the monomials are a conditional basis sequence with respect to the norm  $M_p(\cdot, 1)$ . So we find  $N \subset \mathbb{N}$  and  $y_n \in Y_n := \text{span} \{z^j : 0 \leq j \leq m_n\}$  with  $1 = M_p(y_n, 1) = 1 \leq nM_p(T_N y_n)$ . Find  $k_n$  with  $b_{k_n} - a_{k_n} > m_n$ , put  $Y_n = \{z^j : a_{k_n} \leq j \leq b_{k_n}\} \subset A_\mu^p$  and define  $S_n : X_n \rightarrow Y_n$  by

$$(S_n f)(z) = z^{a_{k_n}} f(z/s_n).$$

Then, according to our assumptions we have  $\|S_n\| \cdot \|S_n^{-1}\| \leq d_n/c_n < c$  for some universal constant  $c$ . Put  $M_n = \{a_{k_n} + j : j \in N, j \leq m_n\}$ . Then  $S_n T_N S_n^{-1} = T_{M_n}|_{X_n}$ . If we consider  $M = \cup_n M_n$  then the preceding shows that  $T_M$  is unbounded on  $A_\mu^p$ . This proves that the system of monomials is conditional in  $A_\mu^p$ .  $\square$

**Conclusion of the proof of Theorem 4.1.** (i)  $\Leftrightarrow$  (ii) follows from the definition of solid hull while (ii)  $\Leftrightarrow$  (iii) follows from the definition of solid core. (Recall, in any case the monomials are a basis of  $A_\mu^p$ .) Now (iii) and Lemma 4.5 imply (v). Finally, (v) and (2.1) imply (iv) while (iv) trivially implies (iii).  $\square$

**Proof of Corollary 4.3.** Proposition 3.5 of [8] shows that, for  $R = 1$ , the assumptions of Lemma 4.5 are always satisfied. Hence the system of monomials can never be unconditional. In view of Theorem 4.1 the Bergman space  $A_\mu^p$  can never be solid.  $\square$

## 5. SOLID WEIGHTED SPACES OF ENTIRE FUNCTIONS WITH SUP-NORMS.

In this section we consider weighted Banach spaces of analytic functions with sup-norms. The main result Theorem 5.2. of this section complements Theorem 4.1. This result was announced in Remark 5.6 of [3]. Here, as in section 3, a continuous weight  $v : \mathbb{C} \rightarrow ]0, \infty[$  is a function satisfying

$$\begin{aligned} v(z) &= v(|z|), z \in \mathbb{C}, \quad v(r) \geq v(s) \text{ if } 0 \leq r < s \\ \text{and } \lim_{r \rightarrow \infty} r^n v(r) &= 0 \text{ for all } n \geq 0. \end{aligned}$$

We deal with the weighted space  $H_v^\infty$  over  $\mathbb{C}$ , i.e.

$$H_v^\infty = \{f : \mathbb{C} \rightarrow \mathbb{C} : f \text{ holomorphic}, \|f\|_v := \sup_{z \in \mathbb{C}} |f(z)|v(z) < \infty\}.$$

Let  $H_v^0$  be the closure of the polynomials in  $H_v^\infty$ .

Similarly to the weighted  $L_p$ -norms in section 3 and 4 one sees that it suffices to require only  $v(r) \geq v(s)$  for  $r_0 \leq r < s$  and some  $r_0 > 0$  since  $\|f\|_v$  and  $\sup_{r_0 \leq |z| < \infty} |f(z)|v(z)$  are equivalent for holomorphic  $f$ .

Again, for  $n > 0$  let  $r_n \in [0, \infty[$  be a point where the function  $r \mapsto r^n v(r)$  attains its global maximum. The next lemma can be easily proved with induction (which was done in [9], Lemma 5.1.). The indices  $m_n$  are needed in the following.

**Lemma 5.1.** *For any  $b > 2$  there are numbers  $0 < m_1 < m_2 < \dots$  with  $\lim_{n \rightarrow \infty} m_n = \infty$  and*

$$b = \min \left( \left( \frac{r_{m_n}}{r_{m_{n+1}}} \right)^{m_n} \frac{v(r_{m_n})}{v(r_{m_{n+1}})}, \left( \frac{r_{m_{n+1}}}{r_{m_n}} \right)^{m_{n+1}} \frac{v(r_{m_{n+1}})}{v(r_{m_n})} \right).$$

Actually, one can show that Lemma 5.1. works for all  $b > 1$  but we need  $b > 2$  in the following proof.

There are examples of weights on  $\mathbb{C}$  such that the monomials  $(z^n)_{n=0}^\infty$  are a Schauder basis in the Banach space  $H_v^0$ . This is the same as saying that the Taylor

series of each element in  $H_v^0$  converges with respect to the weighted sup-norm  $\|\cdot\|_v$ . In the known examples, in this case,  $(z^n/\|z^n\|_v)_{n=0}^\infty$  is equivalent to the unit vector basis of  $c_0$ . Moreover, here  $H_v^\infty$  is solid. We show that this is always true provided that  $(z^n)_{n=0}^\infty$  is a Schauder basis of  $H_v^0$ . We also characterize this situation by a property for the indices  $m_n$  of Lemma 5.1. Our arguments are similar to those of [8].

Let  $h(z) = \sum_{k=0}^\infty b_k z^k$ . As before let  $P_n$  be the partial sum operators, i.e.

$$(P_n h)(z) = \sum_{k=0}^n b_k z^k.$$

If the monomials are a basis of  $H_v^0$  then  $\sup_n \|P_n|_{H_v^0}\| = \sup_n \|P_n|_{H_v^\infty}\| < \infty$ .

For any  $k$  we have

$$(5.1) \quad |b_k| \cdot \|z^k\|_v = |b_k| r_k^k v(r_k) = \left| \frac{1}{2\pi} \int_0^{2\pi} h(r_k e^{i\varphi}) e^{-ik\varphi} d\varphi \right| v(r_k) \leq \|h\|_v.$$

Moreover take the numbers  $m_n$  of Lemma 5.1. and put

$$(R_n h)(z) = \sum_{k=0}^{m_n-1} b_k z^k + \sum_{m_{n-1} < k \leq m_n} \frac{[m_n] - k}{[m_n] - [m_{n-1}]} b_k z^k.$$

Finally put  $M_\infty(h, r) = \sup_{|z|=r} |h(z)|$ .

**Theorem 5.2.** *The following are equivalent*

- (i)  $\sup_n (m_{n+1} - m_n) < \infty$  where  $m_n$  are the indices of Lemma 5.1.
- (ii)  $(z^n)_{n=0}^\infty$  is a Schauder basis of  $H_v^0$ .
- (iii)  $(z^n/\|z^n\|_v)_{n=0}^\infty$  is equivalent to the unit vector basis of  $c_0$ .
- (iv)  $H_v^\infty$  is solid.
- (v)  $H_v^0$  is solid.

*Proof.* Put  $V_n = R_n - R_{n-1}$ . According to Proposition 5.2 in [9], since we assumed  $b > 2$  in Lemma 5.1., the norms  $\|h\|_v$  and  $\sup_n \sup_{r_{m_{n-1}} \leq r \leq r_{m_{n+1}}} M_\infty(V_n h, r) v(r)$  are equivalent. Since Lemma 3.3 in [9] implies that the operators  $V_n$  are uniformly bounded on  $H_v^\infty$ , we obtain constants  $c_1 > 0$  and  $c_2 > 0$  with

$$(5.2) \quad c_1 \sup_n \|V_n h\|_v \leq \|h\|_v \leq c_2 \sup_n \|V_n h\|_v \quad \text{for all } h \in H_v^\infty.$$

(i)  $\Rightarrow$  (ii): Observe that, by definition of  $V_n$ ,  $\dim V_n(H_v^0) = [m_{n+1}] - [m_{n-1}]$ . By (i) we obtain  $\sup_n \dim V_n(H_v^0) < \infty$ . With the definition of  $P_j$  and (5.1) we see that  $\sup_{j,n} \|P_j|_{V_n(H_v^0)}\| \leq \sup_n ([m_{n+1}] - [m_{n-1}]) < \infty$ . With (5.2) and  $P_j V_n = V_n P_j$  for all  $j$  and  $n$  we conclude that the projections  $P_j$  are uniformly bounded. Hence  $(z^n)_{n=0}^\infty$  is a Schauder basis of  $H_v^0$ .

(ii)  $\Rightarrow$  (i): Assume that (ii) holds. By definition,  $V_n(P_{m_{n+1}} - P_{m_{n-1}}) = V_n$ . In view of the uniform boundedness of the  $V_n$  and (5.2) we obtain constants  $c'_1 > 0$  and  $c'_2 > 0$  with

$$(5.3) \quad c'_1 \sup_n \|(P_{m_{n+1}} - P_{m_n})h\|_v \leq \|h\|_v \leq c'_2 \sup_n \|(P_{m_{n+1}} - P_{m_n})h\|_v$$

for all  $h \in H_v^\infty$ . Here the first inequality follows from the uniform boundedness of the  $P_n$  in view of (ii) while the second inequality follows from (5.2). Let  $t_n \in [0, R[$

be such that

$$t_n = r_{m_n} \quad \text{if} \quad b = \left( \frac{r_{m_{n+1}}}{r_{m_n}} \right)^{m_{n+1}} \frac{v(r_{m_{n+1}})}{v(r_{m_n})}$$

and

$$t_n = r_{m_{n+1}} \quad \text{if} \quad b = \left( \frac{r_{m_n}}{r_{m_{n+1}}} \right)^{m_n} \frac{v(r_{m_n})}{v(r_{m_{n+1}})}$$

in Lemma 5.1. Then Corollary 3.2.(b) of [9] implies

$$\|(P_{m_{n+1}} - P_{m_n})h\|_v \leq 2bM_\infty((P_{m_{n+1}} - P_{m_n})h, t_n)v(t_n).$$

With (5.3) we obtain

$$(5.4) \quad d_1 \sup_n M_\infty((P_{m_{n+1}} - P_{m_n})h, t_n)v(t_n) \leq \|h\|_v \leq d_2 \sup_n M_\infty((P_{m_{n+1}} - P_{m_n})h, t_n)v(t_n)$$

for some constants  $d_1 > 0$ ,  $d_2 > 0$  and all  $h \in H_v^0$ .

It is well-known that there are bounded holomorphic functions whose Taylor series do not converge with respect to  $M_\infty(\cdot, 1)$ . By going over to suitable Cesaro means if necessary, we see that, for each  $n \in \mathbb{N}$ , there is a polynomial  $f$  of degree  $N$  and an index  $M \leq N$  such that

$$M_\infty(f, 1) = 1 \quad \text{but} \quad n \leq M_\infty(P_M f, 1).$$

Proceeding by contradiction, assume that (i) does not hold, that is  $\sup_n(m_{n+1} - m_n) = \infty$ . Then we find  $k$  with  $\dim(P_{m_{k+1}} - P_{m_k})H_v^0 > N$ . Put  $h(z) = z^{m_k} f(z)/v(t_k)$ . Then, in view of (5.4), we obtain

$$d_1 \leq \|h\|_v \leq d_2 \quad \text{and} \quad \frac{n}{d_2} \leq \|P_{M+m_k} h\|_v.$$

This implies that the projections  $P_j$  are not uniformly bounded contradicting the assumption (ii). This contradiction implies  $\sup_n(m_{n+1} - m_n) < \infty$ , and we have checked that (ii)  $\Rightarrow$  (i).

Moreover, if  $\sup_n(m_{n+1} - m_n) < \infty$  then (5.4) easily implies that the normalized monomials are equivalent to the unit vector basis of  $c_0$ . Hence we have (ii)  $\Rightarrow$  (iii). (iii)  $\Rightarrow$  (ii) is trivial.

(iii)  $\Rightarrow$  (iv): By the preceding we know already that (iii) implies (ii) and hence (5.4). If  $\sigma_n$  is the  $n$ 'th Cesaro mean and  $h \in H_v^\infty$  then  $\sigma_n h \in H_v^0$ . We have  $\sigma_n P_j = P_j \sigma_n$  for all  $n$  and  $j$ . Moreover  $\|\sigma_n h\|_v \leq \|h\|_v$  and  $\sup_n \|\sigma_n h\|_v = \|h\|_v$ . This implies that (5.4) remains valid for all  $h \in H_v^\infty$ . This together with the fact that  $\sup_n(m_{n+1} - m_n) < \infty$  shows that  $H_v^\infty$  is solid.

(iv)  $\Rightarrow$  (iii) follows from Theorem 5.2 in [3].

(iv)  $\Rightarrow$  (v): If  $g \in S(H_v^0)$  then, by definition and (iii),

$$\lim_{n \rightarrow \infty} \hat{g}(n) \|z^n\|_v = 0$$

which implies by (iii) that  $g \in H_v^0$ .

(v)  $\Rightarrow$  (iv): If  $g \in S(H_v^\infty)$  then by definition  $\sigma_n g \in S(H_v^0) = H_v^0$  for all  $n$ . This implies  $g \in H_v^\infty$ .  $\square$

In [9] it was shown that  $v(r) = \exp(-\log^2(r))$ ,  $R = \infty$ , satisfies (iv) (and hence all assertions) of Theorem 5.2.

Observe that nowhere in the preceding proof the fact that our functions are defined on  $\mathbb{C}$  is used. The arguments work as well for weighted spaces of holomorphic

functions over the unit disc  $\mathbb{D}$ . However in this case  $\lim_{n \rightarrow \infty} r_n = 1$  and this fact together with

$$4 < b^2 \leq \left( \frac{r_{m_{n+1}}}{r_{m_n}} \right)^{m_{n+1} - m_n}$$

(by Lemma 5.1.) implies  $\sup_n (m_{n+1} - m_n) = \infty$ . This means that in the case of holomorphic functions over  $\mathbb{D}$  the preceding theorem is empty. Compare with Corollary 5.3 in [3].

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